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$$u_\lambda=\begin{vmatrix} v_{\lambda-1} & v'_{\lambda-1} \\ u_{\lambda-1} & u'_{\lambda-1} \end{vmatrix}, \text{ and } u'_\lambda=\begin{vmatrix} v_{\lambda-1} & v''_{\lambda-1} \\ u_{\lambda-1} & u''_{\lambda-1} \end{vmatrix} \cdots$$

Also solved by G. B. M. Zerr.

267. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

A point within an ellipse, upon a normal making an angle λ with the major axis, is arbitrarily chosen. With this point as pole, and the line through it parallel to the major axis as polar axis, the equation of the ellipse is, $A\cos^4\theta+B\cos^3\theta+C\cos^2\theta+D\cos\theta+E=0$, where the coefficients are functions of λ, of the radius vector ρ, and of the distance along the normal to the pole, ρ_1. Evidently for $\rho=\rho_1$, a solution is $\cos\theta=\cos\lambda$. Required the multiplicity of this solution for any values of ρ_1, $[\lambda\neq 0,\ \rho_1\neq 0]$.

Solution by the PROPOSER.

I. *Analytical Solution.* The normal to $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1$ at (x_0, y_0) is, denoting its slope by $\tan\lambda$, $\dfrac{x_0-\xi}{\cos\lambda}=\dfrac{y_0-\eta}{\sin\lambda}=r$, where r is the distance from (x_0, y_0) to any point (ξ, η), being positive for interior points. Now, $\tan\lambda=-\dfrac{dx_0}{dy_0}=\dfrac{x_0}{x_0(1-e^2)}$, whence $x_0^2+x_0^2(1-e^2)\tan^2\lambda=a^2$, or

$$(1)\qquad x_0=\mu\cos\lambda,\ y_0=\mu\sin\lambda(1-e^2),\ \mu=\frac{a}{\sqrt{(1-e^2\sin^2\lambda)}}.$$

And, if the pole (x_1, y_1):

$$(2)\qquad x_1=x_0-\rho_1\cos\lambda,\ y_1=y_0-\rho_1\sin\lambda.$$

Transfer the origin to (x_1, y_1) and introduce polar coordinates by $x=x_1+\rho\cos\theta$, $y=y_1+\rho\sin\theta$; the ellipse becomes

$$(3)\qquad (1-e^2)(x_1+\rho\cos\theta)^2+(y_1+\rho\sin\theta)^2=a^2(1-e^2).$$

Collect, and eliminate $\sin\theta$:

$$(4)\qquad (\alpha\cos^2\theta+\beta\cos\theta+\gamma)^2=(-\delta\sin\theta)^2=\delta^2(1-\cos^2\theta),$$

$$(5)\qquad \text{where } \alpha=-e^2\rho^2,\ \beta=2x_1\rho(1-e^2),\ \delta=2y_1\rho,$$
$$\text{and } \gamma=\rho^2+(x_1^2-a^2)(1-e^2)+y_1^2.$$

The re-arrangement of (4) in descending powers of $\cos\theta$ gives the form of equation used in stating the problem; the above form will, however, suffice.

When $\rho=\rho_1$ let the coefficients be denoted by $\alpha_1, \ldots, \delta_1$; and let $F(\cos\theta)\equiv(\alpha_1\cos^2\theta+\beta_1\cos\theta+\gamma_1)^2-\delta_1^2(1-\cos^2\theta)$. Now, from the derivative of (4), it is obvious that $\cos\theta=\cos\lambda$ is a solution when $\rho=\rho_1$; *i. e.*, $F(\cos\lambda)=0$. The multiplicity of this solution will be shown by the derivatives. Differentiating with respect to $\cos\theta$, and substituting $\cos\lambda$:

$$\text{(6)}\qquad \begin{aligned} F'(\cos\lambda) &\equiv 2(\alpha_1\cos^2\lambda+\beta_1\cos\lambda+\gamma)(2\alpha_1\cos\lambda+\beta_1)+2\delta_1^2\cos\lambda \\ &= -2\delta_1\sin\lambda(2\alpha_1\cos\lambda+\beta_1)+2\delta_1^2\cos\lambda \end{aligned}$$

because of (4). Substituting by (5), (2) and (1):

$$F'(\cos\lambda)=4\delta_1\rho_1\sin\lambda\cos\lambda[e^2\rho_1-(\mu-\rho_1)(1-e^2)+\mu(1-e^2)-\rho_1]\equiv 0.$$

$$\text{(7) Again.}\quad \begin{aligned} F''(\cos\lambda) &\equiv 4\alpha_1(\alpha_1\cos^2\lambda+\beta_1\cos\lambda+\gamma_1)+2(2\alpha_1\cos\lambda+\beta_1)^2+2\delta_1^2 \\ &\equiv -4\alpha_1\delta_1\sin\lambda+2\delta_1^2\cot^2\lambda+2\delta_1^2, \end{aligned}$$

because of (4) and (6). Substituting by (5), (2) and (1):

$$\text{(8)}\qquad F''(\cos\lambda)=\frac{4\rho_1^2y_1}{\sin\lambda}[\mu(1-e^2)-\rho_1(1-e^2\sin^2\gamma)].$$

This vanishes if, and only if, $y_1=0$ or $\rho_1=\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$.

$$\text{(9)}\qquad \text{Again, } F'''(\cos\lambda)\equiv 12\alpha_1[2\alpha_1\cos\lambda+\beta_1]=\frac{12\alpha_1\delta_1\cos\lambda}{\sin\lambda}.$$

This vanishes if, and only if, $y_1=0$ or $\cos\lambda=0$.

Geometrically, these results show that, with the pole at any point on the normal, the solution $\cos\theta=\cos\lambda$ is of order at least two; with the pole at the center of curvature for the given normal, the solution is of order three; with the pole on the major axis, of order four.

[That $\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$ is the radius of curvature is seen thus: the usual expression, $R=\dfrac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, reduces immediately by using (1) and $b^2=a^2(1-e^2)$ to $\dfrac{\mu^3(1-e^2)}{a^2}$ or $\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$.]

The vanishing of $F'''(\cos\lambda)$ when $\cos\lambda=0$ is unimportant, since $F''(\cos\lambda)\neq 0$ when $\cos\lambda=0$.

II. *Geometrical Solution.* Geometrically, $\rho=\rho_1$ is the equation of a circle, of radius ρ_1, with its center at the pole. The order of the solution

$\cos\theta = \cos\lambda$ of the given equation is at least as great as the number of coincident points in which the circle cuts the ellipse at the given point. [How the order may be greater than this number will appear shortly.]

The pole being any point whatever on the normal, the circle is tangent to the ellipse, since the two curves have a common normal. There are consequently at least two coincident points of intersection.

If the pole be at the center of curvature for the given normal, the circle is the osculating circle, which cuts the ellipse in at least three coincident points.

If the pole be on the major axis, then it follows from the symmetry that the circle is tangent to the ellipse at two points, where $\theta=\lambda$ and where $\theta=-\lambda$. But since $\cos(-\lambda)=\cos\lambda$, it follows that for all four intersections $\cos\theta=\cos\lambda$, so that this solution is of multiplicity four.

MECHANICS.

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220. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Four particles A, B, C, D, lie on a smooth table at the corners of a rhombus. AB, BC, CD, DA are light inextensible strings connecting the particles. The angle at A is acute. A blow is given to A along the diagonal, away from C. Find the ratio of the initial velocity of C to that of A.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let v=velocity of A. Then since the table and particles are smooth the component of v along the direction BA is $y=v\cos\frac{1}{2}A$. The component of y in the direction CB is $z=y\cos A=v\cos A\cos\frac{1}{2}A$. The component of z in the direction CA is $u=z\cos\frac{1}{2}A$.

$\therefore u=v\cos A\cos^2\frac{1}{2}A$. But u is the initial velocity of C.

$\therefore v : u=1 : \cos A\cos^2\frac{1}{2}A$.

As $\angle A$ decreases every instant, u increases every instant until $\angle A=0^\circ$, $u=v$.

221. Proposed by W. J. GREENSTREET, Stroud, England.

Two smooth intersecting planes are each at 45° to the horizon. Between them lies a cylinder of elliptic cross section. Find the position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let P, Q be the points of contact of the cylinder with the planes, intersecting at an angle of 90°. Let the normals at P, Q intersect in O and let C be the center of an elliptic section in the same plane as OP, OQ. Then either C and O coincide or CO is vertical. If they coincide the axes of the elliptic section are parallel to the planes. This gives one position of equilibrium. When OC is vertical, OC makes an angle of 45° with both planes.